The Distant-l Chromatic Number of Random Geometric Graphs

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Abstract

A random geometric graph G_n is given by picking n vertices in $\mathbb{R}^d$ independently under a common bounded probability distribution, with two vertices adjacent if and only if their l^p-distance is at most r_n. We investigate the distant-l chromatic number $\chi_l(G_n)$ of G_n for $l \geq 1$. Complete picture of the ratios of $\chi_l(G_n)$ to the chromatic number $\chi(G_n)$ are given in the sense of almost sure convergence.

Keywords: chromatic number; distance coloring; random geometric graph.

1. Introduction

We consider in this short paper the distance coloring of random geometric graph $G_n := G(\mathcal{X}_n, r_n)$, which is obtained as follows. Take l^p-norm $||\cdot||$ in $\mathbb{R}^d$ for $1 \leq p \leq \infty$. Let f be some bounded probability density function on $\mathbb{R}^d$ and let $\mathcal{X}_n = \{X_1, X_2, \cdots, X_n\}$, where $\{X_i\}$ are i.i.d. random d-vectors with the common density f. Let r_n be a sequence of distance satisfying $r_n \to 0$ as $n \to \infty$. Then we denote by $G(\mathcal{X}_n, r_n)$ the graph with vertex set $\mathcal{X}_n$ and with edges X_iX_j, if $||X_i - X_j|| < r_n$ for all $i \neq j$. An excellent introduction to random geometric graphs is available in [11]. The vertex coloring in geometric graphs is closely related with the *radio channel assignment problem*, see e.g. [3, 6, 7] for its history as well as an extensive treatment of this significant issue.

Recall that a k-coloring of a graph G is a map $g : V(G) \to \{1, 2, \cdots, k\}$ such that $g(u) \neq g(v)$ whenever $uv \in E(G)$ and that the chromatic number $\chi(G)$ is the least k for which G is k-colorable. For a graph G, the graph distance $d_G(u, v)$ between two vertices u and v is defined as the length of a shortest path joining them (hence the graph distance will be infinity if they are in different components). Thereby we get two kinds of distance between two vertices in our geometric setting, i.e. $d_G(u, v)$ and $||u - v||$, whose interrelationship (Lemma 4) turns to be important in the proof of our main theorems. For $l \geq 1$, a distant-l coloring of G is a coloring of the vertices such that vertices at distance (d_G) less than or equal to l have different colors. The least number for which a distant-l coloring exists is called the *distant-l chromatic number* of G, designated by $\chi_l(G)$. Recall that a distant-l coloring of G is equivalent to an ordinary vertex coloring of l power G^l of G. (The l power of a graph G, denoted as G^l, is the graph with the same vertex set and in which two vertices are joined by an edge if and only if they have distance (d_G) less than or equal to l in G.) Hence $\chi(G^l) = \chi_l(G)$, particularly, $\chi_1(G) = \chi(G)$.

Distance coloring has been a long standing topic in graph theory and has been dealt with mostly in planar graphs. We refer the reader to [1, 4, 5] and [9] for more details regarding this subject. Recently, Díaz et al. [2] studied the distant-2 chromatic number of G_n in the Euclidean plane by takeing $f = 1_{[0,1]^2}$. Their results show that the order of $\chi_2(G_n)$ is consistent with that of $\chi(G_n)$ in the connectivity regime, i.e. when $nr_n^2 = \Theta(\ln n)$. In the study of random geometric graph G_n, some limiting regimes for r_n are of special interest [11]. One of these is connectivity regime in which $nr_n^d = \Theta(\ln n)$ as

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mentioned above. When $nr_n^d \gg \ln n$ we refer to the limiting regime as the superconnectivity regime and the limiting regime $nr_n^d \ll \ln n$ is called the subconnectivity regime. Our aim, in this paper, is to determine the strong law results of the ratios of $\chi_l(G_n)$ to $\chi(G_n)$ in all the above three cases. In addition, a focusing phenomenon that the probability measure becomes concentrated on two consecutive integers is observed in [10] for $\chi(G_n)$ in the subconnectivity regime. We shall state an analogous result (Theorem 1) for $\chi_l(G_n)$, whose proof may be adapted from [10] straightforwardly.

Before going further, we introduce some preliminary definitions. Let vol$(\cdot)$ denote the d-dimensional Lebesgue measure. In the rest of the paper, let $f_{\max} := \sup\{t|\mathrm{vol}(\{f(x) > t\}) > 0\}$ be the essential supremum of the probability density function f. For any graph G, we denote the maximum degree of G by $\triangle(G)$ and the clique number of G by $\omega(G)$. Recall that we have the basic inequalities: $\omega(G) \le \chi(G) \le \triangle(G) + 1$. Given $x \in \mathbb{R}^d$ and $r > 0$, let $B(x, r)$ be the ball centered at x with radius r. We will say that a sequence of events E_n holds a.s. (almost surely) if $P(E_n) \to 1$ as $n \to \infty$.

The rest of this paper is organized as follows. In Section 2, we give our main results for the distant chromatic number. Section 3 contains the proofs. We conclude the paper in Section 4.

2. Statement of main results

Throughout the paper, we assume the probability density function f is bounded, that is, $f_{\max} < \infty$. We alluded to the following result in Section 1.

Theorem 1.(Focusing) *If $nr_n^d = o(\ln n)$ and $l \ge 1$ fixed, then there exists a sequence $\{a_n\}$ such that*

$$P(a_n \le \chi_l(G_n) \le a_n + 1) \to 1, \qquad as \quad n \to \infty.$$

We leave the proof as an exercise for the readers consulting Corollary 2 of [10].

Let supp$f$ be the support of f, i.e. supp$f := \overline{\{x|f(x) > 0\}}$. Let f_0 be the essential infimum of f over supp$f$, that is, the largest h such that $P(f(X_1) \ge h) = 1$.

Theorem 2.(Superconnectivity regime) *Suppose $nr_n^d \gg \ln n$ and $l \ge 1$ fixed. Let f satisfies (a) a.e. continuous and* supp$f$ *does not contain a sequence of isolated points which has a limit point in* supp$f$*; or (b) $f_0 > 0$, then*

$$\frac{\chi_l(G_n)}{l^d \chi(G_n)} \to 1 \qquad a.s.$$

We remark that the conditions (a) and (b) imposed on f above are rather mild; in fact, typical distributions such as normal distribution and $f = 1_{[0,1]^d}$ are clearly allowed.

Theorem 3.(Connectivity regime) *Suppose $nr_n^d = \Theta(\ln n)$ and $l \ge 1$ fixed, then*

$$\frac{\chi_l(G_n)}{l^d \chi(G_n)} \to c \qquad a.s.$$

where $c \in [l^{-d}, 1]$ is a constant depending only on the quantity involved in "Θ".

The constant c will be explicitly given in the proof.

Theorem 4.(Subconnectivity regime) *Suppose $n^{-\varepsilon} \ll nr_n^d \ll \ln n$ for all $\varepsilon > 0$ and $l \ge 1$ fixed, then*

$$\frac{\chi_l(G_n)}{\chi(G_n)} \to 1 \qquad a.s.$$

Notice that Theorem 2 and 3 rely explicitly on the dimension of the underlying space $\mathbb{R}^d$ while Theorem 4 does not.

To close up the spectrum of limiting regimes, we observe (by exploiting a result in [11] Section 6.1) that if $nr_n^d \le n^{-\varepsilon}$ for some $\varepsilon > 0$, then there exists some $c > 0$ such that $P(\chi_l(G_n) \le c) \to 1$ and $P(\chi(G_n) \le c) \to 1$ as $n \to \infty$. Hence there won't be any interesting strong law in this case.

We refer the readers to [8] for a number of results regarding the strong law of large numbers in chromatic number $\chi(G_n)$, which are largely improved than those discovered earlier by Penrose et al. (see e.g.[11]). Wherefore our theorems suggest the strong laws of $\chi_l(G_n)$.

3. Proofs

We will need some strong law results from [8], which take an important role in the proofs. To make the present work self-contained, some technical definitions are included as follows. For a measurable set $A \subseteq \mathbb{R}^d$, if $\lim_{\varepsilon\to 0} \mathrm{vol}(A_\varepsilon) = \mathrm{vol}(A)$, where $A_\varepsilon := A + B(0, \varepsilon)$, then we say A has a small neighborhood. Let $\mathcal{F}$ be the collection of all non-negative, bounded, measurable functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ with $0 < \mathrm{vol}(\mathrm{supp}\varphi) < \infty$, and $\{x|\varphi(x) > a\}$ having a small neighborhood for all $a \in \mathbb{R}$. Given $\varphi \in \mathcal{F}$, let $M_\varphi := \sup_{x\in\mathbb{R}^d} \sum_{i=1}^n \varphi(\frac{X_i - x}{r_n})$. Set a function $H(x) := x \ln x - x + 1$ for $x > 0$. For $\varphi \in \mathcal{F}$ and $0 < t < \infty$, let us set

$$\xi(\varphi, t) := \int_{\mathbb{R}^d} \varphi(x) e^{s\varphi(x)} \mathrm{d}x,$$

where $s = s(\varphi, t)$ is the unique non-negative solution to the equation $\int_{\mathbb{R}^d} H(e^{s\varphi(x)})\mathrm{d}x = \frac{1}{tf_{\max}}$. We also set $\xi(\varphi, \infty) = \int_{\mathbb{R}^d} \varphi(x)\mathrm{d}x$ naturally. Let $\mathcal{G}$ be the collection of measurable, non-negative functions $\varphi : \mathbb{R}^d \to [0, 1]$ such that $\sum_{x\in A} \varphi(x) \le 1$ for any set $A \subseteq \mathbb{R}^d$ satisfying $||x - y|| > 1$ for all $x \ne y \in A$. Denote $k_n := \frac{\ln n}{\ln(\ln n/nr_n^d)}$ throughout the paper.

Lemma 1 *([8]). Suppose* $\frac{nr_n^d}{\ln n} \to t \in (0, \infty]$, *then*

$$\frac{\chi(G_n)}{nr_n^d} \to f_{\max} \sup_{\varphi\in\mathcal{G}} \xi(\varphi, t) \qquad a.s.$$

Lemma 2 *([8]). Let $W \subseteq \mathbb{R}^d$ be a bounded, measurable set with non-empty interior and having a small neighborhood. Suppose $\varphi = 1_W$. For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $n^{-\delta} \le nr_n^d \le \delta \ln n$ then*

$$P((1 - \varepsilon)k_n \le M_\varphi \le (1 + \varepsilon)k_n \ for\ all\ but\ finitely\ many\ n) = 1,$$

where k_n is defined as above.

We will also need the following property for the functional $\xi(\varphi, t)$.

Lemma 3 *([8]). For $t, h > 0$ and non-negative, bounded, measurable, integrable function φ, we have*

$$\Big(\frac{t}{t + h}\Big)\xi(\varphi, t) \le \xi(\varphi, t + h) \le \xi(\varphi, t).$$

The next lemma reveals that the shortest path between any pair of nodes in G_n is close to a straight line in and above the connectivity regime. This result can be improved, but it will be enough for our purpose here.

Lemma 4. *Suppose $nr_n^d \geq \Theta(\ln n)$ and $l \geq 1$ fixed. Let P_l^n denote the probability $P(d_{G_n}(X_1, X_2) > l, ||X_1 - X_2|| < lr_n)$, then*

$$P_l^n = o(n^{-2}) \qquad \text{as } n \to \infty.$$

Proof. We use the inductive method for l. If $l = 1$, then

$$P_1^n = P(d_{G_n}(X_1, X_2) > 1, ||X_1 - X_2|| < r_n) = 0, \qquad \forall\ n \geq 2.$$

Now assuming that $P_l^n = o(n^{-2})$ for some $l \geq 1$, we aim to prove $P_{l+1}^n = o(n^{-2})$. Assume that $||X_1 - X_2|| = (l + 1 - \varepsilon)r_n$ with $0 < \varepsilon \leq 1$. Let $Q := B(X_1, (l - \varepsilon/2)r_n) \cap B(X_2, r_n)$, see e.g. Fig.1 (which is drawn under l^2-norm and $d = 2$). We have

$$\begin{aligned}
1 - P_{l+1}^n &\geq 1 - P(d_{G_n}(X_1, X_2) > l + 1 \big| \ ||X_1 - X_2|| < (l+1)r_n) \\
&= P(d_{G_n}(X_1, X_2) \leq l + 1 \big| \ ||X_1 - X_2|| < (l+1)r_n) \\
&\geq P(\exists X_i \in Q, d_{G_n}(X_1, X_i) \leq l \big| \ ||X_1 - X_2|| < (l+1)r_n) \\
&= P(\exists X_i \in Q \big| \ ||X_1 - X_2|| < (l+1)r_n) \\
&\quad \cdot P(d_{G_n}(X_1, X_i) \leq l \big| \ \exists X_i \in Q, ||X_1 - X_2|| < (l+1)r_n) \\
&:= P_1(n) \cdot P_2(n) \qquad (1)
\end{aligned}$$

Fig.1 implies that the value of r_n is only a scaling factor and that $\mathrm{vol}(Q)$ is proportional to r_n^d (w.r.t. any l^p-norm $||\cdot||$). Moreover, $\mathrm{vol}(Q) = c_1 r_n^d$ for some positive constant $c_1 = c_1(l, \varepsilon)$ depending only on l and ε. Therefore, by the requirement of probability density f and the asymptotic behavior of r_n, we get

$$\lim_{n\to\infty} (1 - P_1(n)) = \lim_{n\to\infty} \Big(1 - \int_Q f(x)\mathrm{d}x\Big)^n \leq \lim_{n\to\infty} e^{-\mathrm{vol}(Q)c_2} = 0,$$

for some positive constant c_2. Thus $P_1(n) \to 1$ as $n \to \infty$.

On the other hand, by the inductive assumption, $1 - P_2(n) \leq P_l^n = o(n^{-2})$. Hence, $P_2(n) \geq 1 - o(n^{-2})$ as $n \to \infty$. Taking limit in both side of (1) gives

$$P_{l+1}^n = o(n^{-2}) \qquad \text{as } n \to \infty,$$

which concludes the proof. □

Recall that $G_n = G(\mathcal{X}_n, r_n)$ and let $G_n' = G(\mathcal{X}_n, lr_n)$, then it's easy to see that $\chi(G_n') \geq \chi_l(G_n) \geq \chi(G_n)$. Now for $t \in (0, \infty]$, suppose $\frac{nr_n^d}{\ln n} \to t$ as $n \to \infty$. By applying Lemma 1 to graph G_n', and using Lemma 3, we have

$$\frac{\chi(G_n')}{l^d nr_n^d} \sim f_{\max} \sup_{\varphi\in\mathcal{G}} \xi(\varphi, l^d t) \leq f_{\max} \sup_{\varphi\in\mathcal{G}} \xi(\varphi, t) \sim \frac{\chi(G_n)}{nr_n^d} \quad a.s. \qquad (2)$$

Hence $\chi(G_n') \leq l^d \chi(G_n)$ almost surely for large enough n. Therefore we get

$$P(\chi(G_n) \leq \chi_l(G_n) \leq l^d \chi(G_n) \text{ for all but finitely many } n) = 1.$$

Thus it can be seen from Theorem 2 and 4 that the upper bound for $\chi_l(G_n)$ is asymptotically attained in the superconnectivity regime while the lower bound is achieved in the subconnective case.

Proof of Theorem 2. Take $t = \infty$ in (2). From the above discussion and definition of the functional $\xi(\varphi, t)$, we have $\chi(G'_n) = (1+o(1))l^d\chi(G_n)$ almost surely for large enough n. Then it suffices to prove $P(\chi(G'_n) > \chi_l(G_n)) = o(1)$ as $n \to \infty$.

Now we have by Lemma 4,

$$\begin{aligned} P(\chi(G'_n) > \chi_l(G_n)) &\leq P(\ \exists X_i, X_j \ s.t.\ d_{G_n}(X_i, X_j) > l \text{ and } ||X_i - X_j|| < lr_n) \\ &\leq n^2 P_l^n = o(1). \end{aligned}$$

as $n \to \infty$. The proof is then completed. □

Proof of Theorem 3. Suppose $\frac{nr_n^d}{\ln n} \to t \in (0, \infty)$ as $n \to \infty$. Employing Lemma 4 in the same way as the above proof suggests that $\chi_l(G_n) = (1+o(1))\chi(G'_n)$. Therefore by the expression (2),

$$\frac{\chi_l(G_n)}{l^d\chi(G_n)} \sim \frac{\chi(G'_n)}{l^d\chi(G_n)} \to \frac{\sup_{\varphi\in\mathcal{G}} \xi(\varphi, l^d t)}{\sup_{\varphi\in\mathcal{G}} \xi(\varphi, t)} := c(t), \qquad a.s.$$

where, by Lemma 3, $c(t) \in [l^{-d}, 1]$ as claimed. □

Proof of Theorem 4. Observe that

$$\omega(G_n) \leq \chi(G_n) \leq \chi_l(G_n) \leq \triangle(G_n^l) + 1 \leq \triangle(G'_n) + 1.$$

By Lemma 2 and the remarks in [8] (see also [7]), we get almost surely

$$\omega(G_n) \sim k_n, \quad \chi(G_n) \sim k_n, \quad \triangle(G'_n) \sim \frac{\ln n}{\ln(\ln n / l^d n r_n^d)} \sim k_n$$

as $n \to \infty$. Recall that $k_n = \frac{\ln n}{\ln(\ln n/nr_n^d)}$. Also note that k_n tends to infinity by the assumed asymptotic behavior of r_n, which concludes the proof. □

4. Concluding remarks

We have investigated in this paper the asymptotic behavior of $\chi_l(G_n)$ when the parameter l is fixed. It is, however, possible to generalize the results to growing l as long as l dose not increase too quickly. Another issue which we have not studied but might be of significance in practice is the rates of convergence of the ratio $\chi_l(G_n)/\chi(G_n)$.

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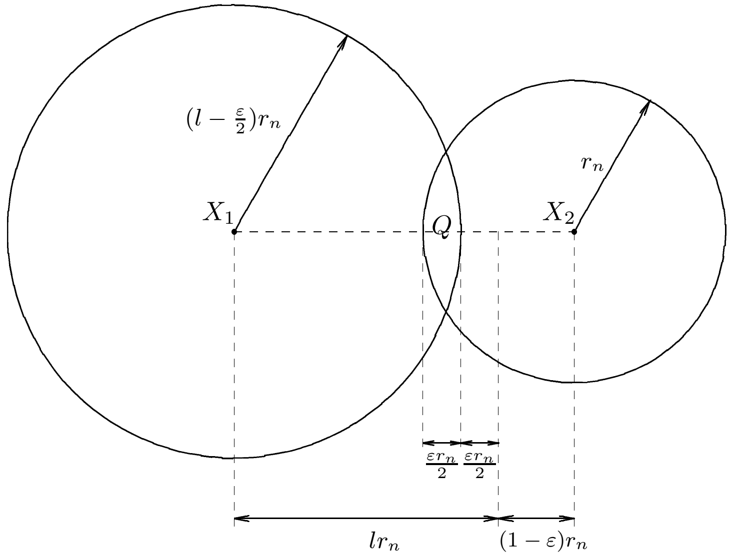

$(l-\frac{\varepsilon}{2})r_n$
r_n
X_1
Q
X_2
$\frac{\varepsilon r_n}{2}$
$\frac{\varepsilon r_n}{2}$
lr_n
$(1-\varepsilon)r_n$

Fig.1